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Associated Jacobi–Laurent polynomials

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Abstract: The Jacobi–Laurent polynomials have been introduced by Hendriksen and van Rossum (1986). In the present paper explicit formulas for the orthogonal Laurent polynomials satisfying the recurrency for the Jacobi–Laurent polynomials with n replaced by $n+b$ are given. These new orthogonal Laurent polynomials are called "associated Jacobi–Laurent polynomials". Using these associated Laurent polynomials, the denominator and the numerator of certain two-point Padé approximants to the pair of functions

$$\frac{z\,F(a,\,b+1;\,c+b+1;\,z)}{F(a,\,b;\,c+b;\,z)} \quad \text{at } 0$$

and

$$\frac{c+b}{-a+b+1}\,\frac{F(-c+1,\,b+1;\,-a+b+2;\,z^{-1})}{F(-c+1,\,b;\,-a+b+1;\,z^{-1})} \quad \text{at } \infty$$

are given. Also some confluent cases are considered.

Keywords: Orthogonal Laurent polynomials, associated Jacobi–Laurent polynomials, two-point Padé approximant, T-fraction.

1. Introduction

In [5] a T-fraction expansion for the quotient

$$\frac{z\,F(a,\,b+1;\,c+1;\,z)}{F(a,\,b;\,c;\,z)} = \phi_0(z)$$

was obtained using one of the contiguous relations for hypergeometric functions. By an equivalence transformation it was shown that this continued fraction is also a T-fraction expansion for

$$\frac{c}{-a+b+1}\,\frac{F(-c+b+1,\,b+1;\,-a+b+2;\,z^{-1})}{F(-c+b+1,\,b;\,-a+b+1;\,z^{-1})} = \phi_\infty(z).$$

The nth approximant, say U_n/V_n, with U_n and V_n polynomials of degree at most n, of this continued fraction, is a two-point Padé approximant to the function

$$\phi(z) = \begin{cases} \phi_0(z) & \text{at } 0, \\ \phi_\infty(z) & \text{at } \infty. \end{cases}$$

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In fact

$$\phi(z) - \frac{U_n(z)}{V_n(z)} = \begin{cases} O(z^{n+1}), & z \to 0, \\ O(z^{-n}), & z \to \infty. \end{cases}$$

Only in the special case $b = 0$, an explicit formula for V_n is given in [5] (with slightly different normalization). From the recurrence relations for the V_n and the Favard theorem it followed that the Laurent polynomials

$$W_{2n}(z) = z^{-n}V_{2n}(z) \quad \text{and} \quad W_{2n+1}(z) = z^{-n-1}V_{2n+1}(z), \qquad n = 0, 1, \ldots,$$

form an orthogonal system of Laurent polynomials. These Laurent polynomials were called "Jacobi–Laurent polynomials".

In the present paper essentially we give closed-form expressions for the "associated Jacobi–Laurent polynomials", i.e., the Laurent polynomials satisfying the recurrence relation for the W_n with n replaced by $n + b$. The associated Jacobi–Laurent polynomials allow us to give explicit formulas for the denominators V_n and the numerators U_n of the nth approximant of the T-fraction expansion of ϕ. The quotient U_n/V_n is a two-point Padé approximant to ϕ. Similar results are obtained for two confluent cases.

In this way we get "Laurent polynomial analogs" of some of the results for ordinary polynomials in [1,8,9]. The approach in the present paper is partially the same as in [9].

In principle we use the notation of the Bateman project [2]. Sometimes we write $F(a, b; c; z)$ as

$$F\left(\begin{matrix} a, b \\ c \end{matrix}\middle| z\right).$$

When it seems convenient the F is written as ${}_2F_1$.

2. Associated Jacobi–Laurent polynomials

In the function ϕ given in the Introduction we may, without loss of generality, replace c by $c + b$. Thus we consider

$$\phi(z) = \begin{cases} \phi_0(z), & |z| \text{ small}, \\ \phi_\infty(z), & |z| \text{ large}, \end{cases}$$

where

$$\phi_0(z) = \frac{z\,F(a, b+1; c+b+1; z)}{F(a, b; c+b; z)}$$

and

$$\phi_\infty = \frac{c+b}{-a+b+1} \frac{F(-c+1, b+1; -a+b+2; z^{-1})}{F(-c+1, b; -a+b+1; z^{-1})}.$$

It follows from the results in [5, Section 3] that we have the formal T-fraction expansions

$$\phi_0(z) \approx \mathop{\mathrm{K}}_{n=1}^{\infty} \frac{f_n z}{1 + g_n z} \quad \text{at } 0 \quad \text{and} \quad \phi_\infty(z) \approx \mathop{\mathrm{K}}_{n=1}^{\infty} \frac{f_n z}{1 + g_n z} \quad \text{at } \infty,$$

where

$$f_1 = 1, \qquad f_n = \frac{(a-c-n-b+1)(n+b-1)}{(c+b+n-2)(c+b+n-1)}, \quad n \geqslant 2,$$

and

$$g_n = \frac{-a+n+b}{c+n+b-1}, \quad n \geqslant 1.$$

The effect of the replacement of c by $c+b$ is that f_n and g_n are now functions of $n+b$. Applying an equivalence transformation on the T-fraction we get

$$\underset{n=1}{\overset{\infty}{\mathrm{K}}} \frac{-B(n+b)z}{z-A(n+b)} \approx \begin{cases} \phi_0(z) & \text{at } 0, \\ \phi_\infty(z) & \text{at } \infty, \end{cases}$$

where

$$B(n+b) = \frac{(n+b-1)(c-a+n+b-1)}{(-a+n+b-1)(-a+n+b)}, \quad n \geqslant 2, \qquad B(1+b) = -\frac{c+b}{-a+b+1}, \tag{2.1}$$

$$A(n+b) = -\frac{c+n+b-1}{-a+n+b}, \quad n \geqslant 1.$$

In order to avoid complications we assume throughout that a, b, c are complex numbers such that

$$a-b, \ -c-b+1, \ -b, \ -c+a-b \notin \mathbb{N}.$$

Then the $A(n+b)$ and the $B(n+b)$ exist and do not vanish. Let

$$\underset{k=1}{\overset{n}{\mathrm{K}}} \frac{-B(k+b)z}{z-A(k+b)} = \frac{U_n(z;\, b)}{V_n(z;\, b)}$$

be the nth approximant of the T-fraction, with U_n and V_n polynomials in z, V_n monic. Then

$$U_n(z;\, b) = (z - A(n+b))U_{n-1}(z;\, b) - B(n+b)zU_{n-2}(z;\, b), \tag{2.2}$$
$$V_n(z;\, b) = (z - A(n+b))V_{n-1}(z;\, b) - B(n+b)zV_{n-2}(z;\, b), \quad n = 1, 2, \ldots,$$

with initial conditions

$$U_{-1} = 1, \qquad U_0 = 0, \qquad V_{-1} = 0, \qquad V_0 = 1.$$

If in the above we take $b = 0$, then (2.1) reduces to

$$B(n) = \frac{(n-1)(c-a+n-1)}{(-a+n-1)(-a+n)}, \quad n \geqslant 2, \qquad B(1) = -\frac{c}{-a+1},$$

$$A(n) = -\frac{c+n-1}{-a+n}, \quad n \geqslant 1,$$

and we get

$$V_n(z;\, 0) = (z - A(n))V_{n-1}(z;\, 0) - B(n)zV_{n-2}(z;\, 0), \quad n = 1, 2, \ldots,$$

with

$$V_{-1}(z;\, 0) = 0, \qquad V_0(z;\, 0) = 1.$$

This means, see [4,5], that

$$V_n(z;\,0)=\frac{(c)_n}{(-a+1)_n}F(-n,\,-a+1;\,-c-n+1;\,z),\quad n=0,1,\ldots,$$

and

$$W_{2n}(z;\,0)=z^{-n}V_{2n}(z;\,0)\quad\text{and}\quad W_{2n+1}(z;\,0)=z^{-n-1}V_{2n+1}(z;\,0),\qquad n=0,1,\ldots,$$

are the Jacobi–Laurent polynomials as defined in [5] with slightly different parameters.

Since the Laurent polynomials $W_n(z;\,b)$ defined by

$$W_{2n}(z;\,b)=z^{-n}V_{2n}(z;\,b)\quad\text{and}\quad W_{2n+1}(z;\,b)=z^{-n-1}V_{2n+1}(z;\,b),\qquad n=0,1,\ldots,$$

clearly satisfy the recurrence relation for the Jacobi–Laurent polynomials with n replaced by $n+b$, the $W_n(z;\,b)$ are called the "associated Jacobi–Laurent polynomials". By the Favard theorem [5, Theorem 1.1] the sequence $\{W_n(z;\,b)\}_{n=0}^{\infty}$ is orthogonal with respect to a moment functional on the algebra of the Laurent polynomials. Under certain extra conditions on the parameters a, b, c a weight function on the unit circle in $\mathbb{C}$ for the associated Jacobi–Laurent polynomials is given in [3].

Since the $V_n(z;\,b)$ are easier to handle than the $W_n(z;\,b)$, we will mainly work with the $V_n(z;\,b)$. We will usually write the polynomials $V_n(z;\,b)$ and $U_n(z;\,b)$ as V_n and U_n.

Theorem 2.1.

$$V_n(z;\,b)=\frac{(c+b)_n}{(-a+b+1)_n}\sum_{k=0}^{n}\frac{(-n-b)_k(-a+1)_k}{k!(-c-n-b+1)_k}$$
$$\times{}_4F_3\left(\begin{matrix}b,\,a,\,c+n-k+b,\,-k\\ c+b,\,a-k,\,n-k+b+1\end{matrix}\middle|1\right)z^k,\quad n=0,1,\ldots\,. \tag{2.3}$$

Remark 2.2. If a is a nonnegative integer, (2.3) should be understood by continuity.

In the proof of Theorem 2.1 we need two lemmas.

Lemma 2.3.

$$\frac{\beta(\gamma-\alpha)}{\gamma}F\left(\begin{matrix}\alpha,\,\beta+1\\ \gamma+1\end{matrix}\middle|z\right)=\beta\,F\left(\begin{matrix}\alpha,\,\beta\\ \gamma\end{matrix}\middle|z\right)-(1-z)\frac{\mathrm{d}}{\mathrm{d}z}F\left(\begin{matrix}\alpha,\,\beta\\ \gamma\end{matrix}\middle|z\right).$$

Proof. Replacing c by $c+1$ in [2, vol. 1, p.103(42)] yields

$$(c-b)F\left(\begin{matrix}a,\,b\\ c+1\end{matrix}\middle|z\right)+b\,F\left(\begin{matrix}a,\,b+1\\ c+1\end{matrix}\middle|z\right)-c\,F\left(\begin{matrix}a,\,b\\ c\end{matrix}\middle|z\right)=0. \tag{2.4}$$

Replacing b by $b+1$ and c by $c+1$ in [2, vol. 1, p.103(33)] gives

$$(c-a-b)F\left(\begin{matrix}a,\,b+1\\ c+1\end{matrix}\middle|z\right)=(c-b)F\left(\begin{matrix}a,\,b\\ c+1\end{matrix}\middle|z\right)-a(1-z)F\left(\begin{matrix}a+1,\,b+1\\ c+1\end{matrix}\middle|z\right). \tag{2.5}$$

Elimination of $F(a,\ b;\ c+1;\ z)$ from (2.4) and (2.5) gives

$$(c-a)F\left(\begin{matrix}a,\ b+1\\ c+1\end{matrix}\middle| z\right)=c\,F\left(\begin{matrix}a,\ b\\ c\end{matrix}\middle| z\right)-a(1-z)F\left(\begin{matrix}a+1,\ b+1\\ c+1\end{matrix}\middle| z\right). \tag{2.6}$$

Since

$$\frac{\mathrm{d}}{\mathrm{d}z}F\left(\begin{matrix}a,\ b\\ c\end{matrix}\middle| z\right)=\frac{ab}{c}F\left(\begin{matrix}a+1,\ b+1\\ c+1\end{matrix}\middle| z\right),$$

the lemma follows. □

Lemma 2.4.

$$\gamma\,F\left(\begin{matrix}\alpha,\ \beta\\ \gamma\end{matrix}\middle| z\right)=(\gamma-\alpha z)F\left(\begin{matrix}\alpha,\ \beta+1\\ \gamma+1\end{matrix}\middle| z\right)+z(1-z)\frac{\mathrm{d}}{\mathrm{d}z}F\left(\begin{matrix}\alpha,\ \beta+1\\ \gamma+1\end{matrix}\middle| z\right).$$

Proof. Replacing $a,\ b,\ c$ by $a+1,\ b+1,\ c+2$ in [2, vol. 1, p.103(42)] gives

$$(c-b)F\left(\begin{matrix}a+1,\ b+1\\ c+2\end{matrix}\middle| z\right)=(c+1)F\left(\begin{matrix}a+1,\ b+1\\ c+1\end{matrix}\middle| z\right)-(b+1)F\left(\begin{matrix}a+1,\ b+2\\ c+2\end{matrix}\middle| z\right). \tag{2.7}$$

Replacing $a,\ b,\ c$ by $a+1,\ b+1,\ c+1$ in [2, vol. 1, p.103(38)] gives

$$\begin{aligned}(c-b)z\,F\left(\begin{matrix}a+1,\ b+1\\ c+2\end{matrix}\middle| z\right)&=(c+1)F\left(\begin{matrix}a,\ b+1\\ c+1\end{matrix}\middle| z\right)\\&\quad-(c+1)(1-z)F\left(\begin{matrix}a+1,\ b+1\\ c+1\end{matrix}\middle| z\right).\end{aligned} \tag{2.8}$$

Elimination of $F(a+1,\ b+1;\ c+2;\ z)$ from (2.7) and (2.8) gives

$$(c+1)F\left(\begin{matrix}a,\ b+1\\ c+1\end{matrix}\middle| z\right)=(c+1)F\left(\begin{matrix}a+1,\ b+1\\ c+1\end{matrix}\middle| z\right)-(b+1)zF\left(\begin{matrix}a+1,\ b+2\\ c+2\end{matrix}\middle| z\right). \tag{2.9}$$

Next we eliminate $F(a+1,\ b+1;\ c+1;\ z)$ from (2.9) and (2.6) to obtain

$$\begin{aligned}&(c+1)(c-az)F\left(\begin{matrix}a,\ b+1\\ c+1\end{matrix}\middle| z\right)+a(b+1)z(1-z)F\left(\begin{matrix}a+1,\ b+2\\ c+2\end{matrix}\middle| z\right)\\&\quad-c(c+1)F\left(\begin{matrix}a,\ b\\ c\end{matrix}\middle| z\right)=0.\end{aligned}$$

Since now

$$\frac{\mathrm{d}}{\mathrm{d}z}F\left(\begin{matrix}a,\ b+1\\ c+1\end{matrix}\middle| z\right)=\frac{a(b+1)}{c+1}F\left(\begin{matrix}a+1,\ b+2\\ c+2\end{matrix}\middle| z\right),$$

the lemma follows. □

In the proof of Theorem 2.1 we use moreover

$$F\left(\begin{matrix}\alpha,\ \beta\\ \gamma\end{matrix}\middle| z\right)=\left(1+\frac{-\alpha+\beta+1}{\gamma}z\right)F\left(\begin{matrix}\alpha,\ \beta+1\\ \gamma+1\end{matrix}\middle| z\right)$$
$$+\frac{(\alpha-\gamma-1)(\beta+1)}{\gamma(\gamma+1)}z\,F\left(\begin{matrix}\alpha,\ \beta+2\\ \gamma+2\end{matrix}\middle| z\right). \tag{2.10}$$

Formula (2.10) is one of the nine contiguous relations for F's Padé mentioned in [6].

Proof of Theorem 2.1. Let

$$v_n=v_n(z)=\frac{\Gamma(c+n+b)}{\Gamma(-a+n+b+1)}F\left(\begin{matrix}-a+1,\ -n-b\\ -c-n-b+1\end{matrix}\middle| z\right).$$

Taking $\alpha=-a+1$, $\beta=-n-b$, $\gamma=-c-n-b+1$ in (2.10) it is easily verified that v_n satisfies the recurrence relation

$$X_n=(z-A(n+b))X_{n-1}-B(n+b)zX_{n-2},\quad n=1,2,\ldots, \tag{2.11}$$

with $A(n+b)$ and $B(n+b)$ as in (2.1). So v_n satisfies the same recurrence as V_n.

If $\alpha=c$, $\beta=c-a+n+b-1$, $\gamma=c+n+b-1$, formula (2.10) takes the form

$$F\left(\begin{matrix}c,\ c-a+n+b-1\\ c+n+b-1\end{matrix}\middle| z\right)$$
$$=\left(1+\frac{-a+n+b}{c+n+b-1}z\right)F\left(\begin{matrix}c,\ c-a+n+b\\ c+n+b\end{matrix}\middle| z\right)$$
$$+\frac{(-n-b)(c-a+n+b)}{(c+n+b-1)(c+n+b)}z\,F\left(\begin{matrix}c,\ c-a+n+b+1\\ c+n+b+1\end{matrix}\middle| z\right). \tag{2.12}$$

From (2.12) it is easily seen that

$$w_n=w_n(z)=\frac{\Gamma(-c-n-b)\Gamma(a-n-b)}{\Gamma(-c+a-n-b)\Gamma(-n-b)}z^nF\left(\begin{matrix}c,\ c-a+n+b+1\\ c+n+b+1\end{matrix}\middle| z\right)$$

also satisfies (2.11). In order to prove that v_n and w_n are independent solutions of the difference equation (2.11) it is sufficient to show that

$$\Delta=v_{-1}w_0-v_0w_{-1}$$

is not identically zero. Therefore we also consider

$$w_n^*=w_n^*(z)=z^{c+b}w_n(z).$$

Since v_n and w_n^* are independent solutions of the hypergeometric differential equation

$$z(1-z)y''+[(a+n+b-2)z-(c+n+b-1)]y'-(a-1)(n+b)y=0, \tag{2.13}$$

the Wronskian $W(v_n, w_n^*)$ is not identically zero. It is easily shown that the Wronskian $W(y_1, y_2)$ of any two independent solutions of (2.13) is of the form

$$W(y_1, y_2)=K_nz^{c+n+b-1}(1-z)^{-c+a-1}$$

for some nonzero constant K_n depending on a, b, c and n. Now we use Lemmas 2.3 and 2.4 to relate Δ and $W(v_0, w_0^*)$.

Application of Lemma 2.3 with $\alpha = -a+1$, $\beta = -b$, $\gamma = -c-b+1$ gives

$$\frac{-b(-c+a-b)}{-c-b+1} F\left(\begin{matrix} -a+1, -b+1 \\ -c-b+2 \end{matrix} \middle| z \right) = -b\, F\left(\begin{matrix} -a+1, -b \\ -c-b+1 \end{matrix} \middle| z \right) - (1-z)\frac{\mathrm{d}}{\mathrm{d}z} F\left(\begin{matrix} -a+1, -b \\ -c-b+1 \end{matrix} \middle| z \right),$$

hence

$$\frac{b(c-a+b)}{-a+b} v_{-1} = b v_0 + (1-z) v_0'. \tag{2.14}$$

Application of Lemma 2.4 with $\alpha = c$, $\beta = c-a+b$, $\gamma = c+b$ gives

$$(c+b) F\left(\begin{matrix} c, c-a+b \\ c+b \end{matrix} \middle| z \right) = (c+b-cz) F\left(\begin{matrix} c, c-a+b+1 \\ c+b+1 \end{matrix} \middle| z \right) + z(1-z)\frac{\mathrm{d}}{\mathrm{d}z} F\left(\begin{matrix} c, c-a+b+1 \\ c+b+1 \end{matrix} \middle| z \right)$$

and this implies

$$\frac{b(c-a+b)}{-a+b} z w_{-1} = (c+b-cz) w_0 + z(1-z) w_0'. \tag{2.15}$$

With $w_0 = z^{-c-b} w_0^*$ we obtain from (2.15)

$$\frac{b(c-a+b)}{-a+b} w_{-1} = z^{-c-b}\{ b w_0^* + (1-z)(w_0^*)' \}. \tag{2.16}$$

Combination of (2.14) and (2.16) yields

$$\frac{b(c-a+b)}{-a+b}(v_{-1} w_0 - v_0 w_{-1}) = -z^{-c-b}(1-z) W(v_0, w_0^*),$$

so

$$\Delta = K z^{-1} (1-z)^{-c+a} \tag{2.17}$$

for some nonzero constant K depending on a, b, c.

Hence $\Delta \not\equiv 0$ and v_n and w_n are independent solutions of (2.11) and V_n is a linear combination of v_n and w_n. Clearly

$$V_n(z; b) = \frac{v_{-1}(z) w_n(z) - w_{-1}(z) v_n(z)}{\Delta}. \tag{2.18}$$

The constant K in (2.17) is obtained from the coefficient of z^{-1} in the Laurent series of (2.17) and the representation

$$\Delta = \frac{\Gamma(c+b-1)\Gamma(-c-b)\Gamma(a-b)}{\Gamma(-a+b)\Gamma(-c+a-b)\Gamma(-b)} F\left(\begin{matrix} -a+1, -b+1 \\ -c-b+2 \end{matrix} \middle| z \right) F\left(\begin{matrix} c, c-a+b+1 \\ c+b+1 \end{matrix} \middle| z \right)$$
$$- \frac{\Gamma(c+b)\Gamma(-c-b+1)\Gamma(a-b+1)}{\Gamma(-a+b+1)\Gamma(-c+a-b+1)\Gamma(-b+1)} z^{-1} F\left(\begin{matrix} -a+1, -b \\ -c-b+1 \end{matrix} \middle| z \right) \times F\left(\begin{matrix} c, c-a+b \\ c+b \end{matrix} \middle| z \right).$$

We get

$$K=-\frac{\Gamma(c+b)\Gamma(-c-b+1)\Gamma(a-b+1)}{\Gamma(-a+b+1)\Gamma(-c+a-b+1)\Gamma(-b+1)}. \tag{2.19}$$

So (2.18) can be written as

$$\begin{aligned}\Delta\cdot V_n=&\frac{\Gamma(c+b-1)}{\Gamma(-a+b)}F\left(\begin{matrix}-a+1,\ -b+1\\ -c-b+2\end{matrix}\middle| z\right)\\&\times\frac{\Gamma(-c-n-b)\Gamma(a-n-b)}{\Gamma(-c+a-n-b)\Gamma(-n-b)}z^nF\left(\begin{matrix}c,\ c-a+n+b+1\\ c+n+b+1\end{matrix}\middle| z\right)\\&-\frac{\Gamma(-c-b+1)\Gamma(a-b+1)}{\Gamma(-c+a-b+1)\Gamma(-b+1)}z^{-1}F\left(\begin{matrix}c,\ c-a+b\\ c+b\end{matrix}\middle| z\right)\\&\times\frac{\Gamma(c+n+b)}{\Gamma(-a+n+b+1)}F\left(\begin{matrix}-a+1,\ -n-b\\ -c-n-b+1\end{matrix}\middle| z\right).\end{aligned} \tag{2.20}$$

In (2.20) we apply the Kummer transformation

$$F\left(\begin{matrix}\alpha,\ \beta\\ \gamma\end{matrix}\middle| z\right)=(1-z)^{\gamma-\alpha-\beta}F\left(\begin{matrix}\gamma-\alpha,\ \gamma-\beta\\ \gamma\end{matrix}\middle| z\right),\quad [2,\text{ vol. 1, p.105(2)}]$$

on the F's coming from the w. We get

$$F\left(\begin{matrix}c,\ c-a+n+b+1\\ c+n+b+1\end{matrix}\middle| z\right)=(1-z)^{-c+a}F\left(\begin{matrix}n+b+1,\ a\\ c+n+b+1\end{matrix}\middle| z\right)$$

and a similar expression in the case $n=-1$.

Thus by (2.17) and (2.20)

$$\begin{aligned}Kz^{-1}V_n=&\frac{\Gamma(c+b-1)\Gamma(-c-n-b)\Gamma(a-n-b)}{\Gamma(-a+b)\Gamma(-c+a-n-b)\Gamma(-n-b)}\\&\times z^nF\left(\begin{matrix}-a+1,\ -b+1\\ -c-b+2\end{matrix}\middle| z\right)F\left(\begin{matrix}n+b+1,\ a\\ c+n+b+1\end{matrix}\middle| z\right)\\&-\frac{\Gamma(-c-b+1)\Gamma(a-b+1)\Gamma(c+n+b)}{\Gamma(-c+a-b+1)\Gamma(-b+1)\Gamma(-a+n+b+1)}\\&\times z^{-1}F\left(\begin{matrix}b,\ a\\ c+b\end{matrix}\middle| z\right)F\left(\begin{matrix}-a+1,\ -n-b\\ -c-n-b+1\end{matrix}\middle| z\right).\end{aligned}$$

Now, by (2.19)

$$\begin{aligned}V_n=&\frac{(c+b)_n}{(-a+b+1)_n}F\left(\begin{matrix}-a+1,\ -n-b\\ -c-n-b+1\end{matrix}\middle| z\right)F\left(\begin{matrix}b,\ a\\ c+b\end{matrix}\middle| z\right)\\&-\frac{-a+b}{c+b-1}\cdot\frac{(c-a+b)_{n+1}(b)_{n+1}}{(c+b)_{n+1}(-a+b)_{n+1}}\\&\times z^{n+1}F\left(\begin{matrix}-a+1,\ -b+1\\ -c-b+2\end{matrix}\middle| z\right)F\left(\begin{matrix}n+b+1,\ a\\ c+n+b+1\end{matrix}\middle| z\right),\end{aligned}$$

hence

$$V_n = \frac{(c+b)_n}{(-a+b+1)_n} F\left(\begin{matrix} -a+1, \ -n-b \\ -c-n-b+1 \end{matrix}\middle| z\right) F\left(\begin{matrix} b, \ a \\ c+b \end{matrix}\middle| z\right) + \mathrm{O}(z^{n+1}).$$

In this expression we use [2, vol. 1, p.187(14)] (Cauchy product of ${}_2F_1$-power series),

$$F\left(\begin{matrix} \alpha, \beta \\ \gamma \end{matrix}\middle| pz\right) F\left(\begin{matrix} \alpha', \beta' \\ \gamma' \end{matrix}\middle| qz\right) = \sum_{k=0}^{\infty} \frac{(\alpha)_k(\beta)_k}{k!(\gamma)_k}$$

$$\times {}_4F_3\left(\begin{matrix} \alpha', \beta', 1-\gamma-k, \ -k \\ \gamma', 1-\alpha-k, 1-\beta-k \end{matrix}\middle| \frac{q}{p}\right)(pz)^k$$

with $p = q = 1$, $\alpha = -a+1$, $\beta = -n-b$, $\gamma = -c-n-b+1$, $\alpha' = b$, $\beta' = a$, $\gamma' = c+b$, to obtain

$$V_n(z;\ b) = \frac{(c+b)_n}{(-a+b+1)_n} \sum_{k=0}^{\infty} \frac{(-n-b)_k(-a+1)_k}{k!(-c-n-b+1)_k}$$

$$\times {}_4F_3\left(\begin{matrix} b, \ a, \ c+n-k+b, \ -k \\ c+b, \ a-k, \ n-k+b+1 \end{matrix}\middle| 1\right) z^k + \mathrm{O}(z^{n+1}). \tag{2.21}$$

Since V_n is a polynomial in z of degree n,

$$-\frac{(c+b)_n}{(-a+b+1)_n} \sum_{k=n+1}^{\infty} \frac{(-n-b)_k(-a+1)_k}{k!(-c-n-b+1)_k} {}_4F_3\left(\begin{matrix} b, \ a, \ c+n-k+b, \ -k \\ c+b, \ a-k, \ n-k+b+1 \end{matrix}\middle| 1\right) z^k$$

equals the term $\mathrm{O}(z^{n+1})$ in (2.21), so (2.21) reduces to (2.3). $\square$

Remark 2.5. The ${}_4F_3$ in (2.3) is "Saalschützian". For $k = n$ this ${}_4F_3$ reduces to a (terminating) Saalschützian ${}_3F_2$ with unit argument, which can be summed by [2, vol. 1, p.188(3)]

$${}_3F_2\left(\begin{matrix} \alpha, \ \beta, \ -n \\ \gamma, \ 1+\alpha+\beta-\gamma-n \end{matrix}\middle| 1\right) = \frac{(\gamma-\alpha)_n(\gamma-\beta)_n}{(\gamma)_n(\gamma-\alpha-\beta)_n}.$$

In this way it can be verified that V_n is monic indeed.

Remark 2.6. The associated Jacobi–Laurent polynomials W_n are given by

$$W_{2n}(z;\ b) = z^{-n} V_{2n}(z;\ b) \quad \text{and} \quad W_{2n+1}(z;\ b) = z^{-n-1} V_{2n+1}(z;\ b), \qquad n = 0, 1, \ldots .$$

We have already remarked that the associated Jacobi–Laurent polynomials are orthogonal with respect to a moment functional. Let us denote this moment functional by Φ. Then

$$\Phi\big(W_n(z;\ b) W_k(z;\ b)\big) = 0 \quad \text{if } n \neq k, \ n, \ k = 0, 1, \ldots,$$

$$\Phi\big(W_n(z;\ b)^2\big) \neq 0, \quad n = 0, 1, \ldots .$$

Assumed that Φ is normalized such that $\Phi(1) = B(1+b)$, it follows from (2.2) that

$$\Phi\big(z^{-k} V_n(z;\ b)\big) = \begin{cases} B(1+b)B(2+b) \cdots B(n+1+b) & \text{if } k = 0, \\ 0 & \text{if } k = 1, \ldots, n, \\ (-1)^n \dfrac{B(1+b)B(2+b) \cdots B(n+1+b)}{A(1+b)A(2+b) \cdots A(n+1+b)} & \text{if } k = n+1. \end{cases} \tag{2.22}$$

Now let

$$Y_n(t)=\frac{1}{\Phi(1)}\Phi\left(\frac{V_n(z;\ b)-V_n(t;\ b)}{z-t}\right),\quad n=0,1,\ldots,\tag{2.23}$$

(Φ acts on z). Then using (2.22), it is easily verified that

$$Y_n(t)=(t-A(n+b))Y_{n-1}(t)-B(n+b)tY_{n-2}(t),\quad n=2,3,\ldots,$$

while $Y_0(t)=0$ and $Y_1(t)=1$. As also

$$V_{n-1}(z;\ b+1)=(z-A(n+b))V_{n-2}(z;\ b+1)-B(n+b)z\ V_{n-3}(z;\ b+1),\quad n=2,3,\ldots,\tag{2.24}$$

while $V_{-1}(z;\ b+1)=0$ and $V_0(z;\ b+1)=1$, we must have

$$Y_n(z)=V_{n-1}(z;\ b+1),\quad n=0,1,\ldots.$$

Usually, polynomials defined by a formula like (2.23) are called associated polynomials. Our definition covers the more traditional one. (See also [2, vol. 2, p.162].) If the function ϕ has the following power series developments at 0 and ∞,

$$\phi(z)=\begin{cases}\displaystyle\sum_{n=1}^{\infty}c_nz^n & \text{at } 0,\\ \displaystyle -\sum_{n=0}^{\infty}c_{-n}z^{-n} & \text{at } \infty,\end{cases}$$

then the functional Φ is determined by

$$\Phi(z^n)=c_{-n},\quad n\in\mathbb{Z}.$$

For this reason, ϕ is sometimes called the moment generating function for the orthogonal Laurent polynomials.

3. Two-point Padé approximants

An obvious modification of [5, (2.8)] yields

$$\phi(z)-\frac{U_n(z;\ b)}{V_n(z;\ b)}=\begin{cases}\dfrac{B(1+b)\cdots B(n+1+b)}{(A(1+b)\cdots A(n+b))^2A(n+1+b)}z^{n+1}+\mathrm{O}(z^{n+2}), & z\to 0,\\ -B(1+b)\cdots B(n+1+b)z^{-n}+\mathrm{O}(z^{-n-1}), & z\to\infty.\end{cases}\tag{3.1}$$

This means that $U_n(z;\ b)/V_n(z;\ b)$ is a two-point Padé approximant to ϕ. The denominator polynomials $V_n(z;\ b)$ of this approximant are given by Theorem 2.1. In (2.2) and (2.24) we see that $U_n(z;\ b)$ and $V_{n-1}(z;\ b+1)$ satisfy the same recurrency. Since $U_0(z;\ b)=V_{-1}(z;\ b+1)=0$, $U_1(z;\ b)=-B(1+b)z$ and $V_0(z;\ b+1)=1$, it follows that

$$U_n(z;\ b)=-B(1+b)zV_{n-1}(z;\ b+1),\quad n=0,1,\ldots.\tag{3.2}$$

Since

$$-B(1+b)=\frac{c+b}{-a+b+1},$$

we get

$$U_n(z;\, b)=\frac{(c+b)_n}{(-a+b+1)_n}z\sum_{k=0}^{n-1}\frac{(-n-b)_k(-a+1)_k}{k!(-c-n-b+1)_k}$$
$$\times{}_4F_3\left(\begin{matrix}b+1,\ a,\ c+n-k+b,\ -k\\ c+b+1,\ a-k,\ n-k+b+1\end{matrix}\middle|1\right)z^k. \tag{3.3}$$

In the special case $b=0$ we have

$$U_n(z;\, 0)=\frac{(c)_n}{(-a+1)_n}z\sum_{k=0}^{n-1}\frac{(-n)_k(-a+1)_k}{k!(-c-n+1)_k}{}_4F_3\left(\begin{matrix}1,\ a,\ c+n-k,\ -k\\ c+1,\ a-k,\ n-k+1\end{matrix}\middle|1\right)z^k.$$

Also this result for the numerator polynomial of the two-point Padé approximant to the pair of functions

$$zF(a,\,1;\;c+1;\;z)\qquad \text{at } 0,$$
$$\frac{c}{-a+1}F(-c+1,\,1;\;-a+2;\;z^{-1})\qquad \text{at } \infty,$$

seems to be new.

Let us now consider the question of convergence of the two-point Padé approximants

$$\frac{U_n(z;\, b)}{V_n(z;\, b)}$$

to ϕ. Recall that $U_n(z;\, b)/V_n(z;\, b)$ is the nth approximant of the continued fraction

$$\underset{n=1}{\overset{\infty}{\mathrm{K}}}\frac{-B(n+b)z}{z-A(n+b)}\approx\underset{k=1}{\overset{\infty}{\mathrm{K}}}\frac{f_kz}{1+g_kz}, \tag{3.4}$$

where

$$f_1=1,\qquad f_k=\frac{(a-c-k-b+1)(k+b-1)}{(c+b+k-2)(c+b+k-1)},\quad k\geqslant 2,$$

and

$$g_k=\frac{-a+k+b}{c+k+b-1},\quad k\geqslant 1.$$

Since

$$-\lim_{k\to\infty}f_k=1=\lim_{k\to\infty}g_k,$$

the above T-fractions are limit periodic and it follows from [7] that for any $r\in(0,\,1)$ there is an N such that

$$\underset{k=N+1}{\overset{\infty}{\mathrm{K}}}\frac{f_kz}{1+g_kz} \tag{3.5}$$

converges uniformly in $|z| < r$ to an analytic function h_0 and converges uniformly in $|z| > 1/r$ to an analytic function h_∞. Put

$$h(z) = \begin{cases} h_0(z) & \text{if } |z| < r, \\ h_\infty(z) & \text{if } |z| > \dfrac{1}{r}. \end{cases}$$

From

$$\frac{f_1 z}{|1 + g_1 z} + \cdots + \frac{f_{N-1} z}{|1 + g_{N-1} z} + \frac{f_N z}{|1 + g_N z + w} = \frac{U_N(z;\, b) + U_{N-1}(z;\, b) w}{V_N(z;\, b) + V_{N-1}(z;\, b) w},$$

it follows that the continued fractions in (3.4) are convergent, possible to ∞, and that

$$\mathop{\mathrm{K}}_{k=1}^{\infty} \frac{f_k z}{1 + g_k z} = \frac{U_N(z;\, b) + U_{N-1}(z;\, b) h(z)}{V_N(z;\, b) + V_{N-1}(z;\, b) h(z)}, \quad |z| < r \text{ or } |z| > \frac{1}{r}.$$

Clearly we have $h(0) = 0$, and by an equivalence transformation of (3.5) we see that $h(\infty) = f_{N+1}/g_{N+1}$. It follows that

$$\lim_{z \to 0} \frac{U_N(z;\, b) + U_{N-1}(z;\, b) h(z)}{V_N(z;\, b) + V_{N-1}(z;\, b) h(z)} = 0$$

and

$$\lim_{z \to \infty} \frac{U_N(z;\, b) + U_{N-1}(z;\, b) h(z)}{V_N(z;\, b) + V_{N-1}(z;\, b) h(z)} = \frac{c + b}{-a + b + 1}.$$

This implies that the continued fractions in (3.4) converge uniformly in a neighborhood of 0 and in a neighborhood of ∞. From (3.1) it follows now that the limits must be ϕ. Since $r \in (0, 1)$ is arbitrary and by analytic continuation we see that the continued fractions (3.4) converge to ϕ on $|z| < 1$, $z \neq$ poles of ϕ, and converge to ϕ on $|z| > 1$, $z \neq$ poles of ϕ. So

$$\lim_{n \to \infty} \frac{U_n(z;\, b)}{V_n(z;\, b)} = \phi(z), \quad |z| \neq 1, \; z \neq \text{poles of } \phi.$$

Remark 3.1. It can be shown that

$$\mathop{\mathrm{K}}_{n=1}^{\infty} \frac{f_n z}{1 + g_n z}$$

converges uniformly on every compact set C contained in $|z| \neq 1$ such that C does not contain poles of the meromorphic function

$$\frac{U_N(z;\, b) + U_{N-1}(z;\, b) h(z)}{V_N(z;\, b) + V_{N-1}(z;\, b) h(z)} = \mathop{\mathrm{K}}_{n=1}^{\infty} \frac{f_n z}{1 + g_n z}.$$

4. Confluent cases

Here we consider the following limit cases of the moment generating function ϕ:

(I) $\psi(z) = \lim_{a \to \infty} a\, \phi(a^{-1} z)$,

(II) $\eta(z) = \lim_{c \to \infty} c^{-1} \phi(cz)$.

We give explicit formulas for denominator and numerator of the two-point Padé approximants (of the type of Section 3) to the functions ψ and η. In the special case $b=0$ the denominator polynomials were already implicitly given in [5]. The denominator polynomials derived in the present section are essentially the orthogonal Laurent polynomials associated to the orthogonal Laurent polynomials for the case $b=0$ as mentioned in [5, Section 3b].

Clearly, at least formally, we have

$$\lim_{\alpha\to\infty} F(\alpha,\ \beta;\ \gamma;\ \alpha^{-1}z) = {}_1F_1(\beta;\ \gamma;\ z),\qquad(4.1)$$
$$\lim_{\gamma\to\infty} F(\alpha,\ \beta;\ \gamma;\ \gamma z) = {}_2F_0(\alpha,\ \beta;\ z).$$

Taking $\alpha\to\infty$ respectively $\gamma\to\infty$ in (2.10) we get

$${}_1F_1(\beta;\ \gamma;\ z) = \left(1-\frac{1}{\gamma}z\right){}_1F_1(\beta+1;\ \gamma+1;\ z) + \frac{\beta+1}{\gamma(\gamma+1)}z\,{}_1F_1(\beta+2;\ \gamma+2;\ z)\qquad(4.2)$$

and

$${}_2F_0(\alpha,\ \beta;\ z) = (1+(-\alpha+\beta+1)z)\,{}_2F_0(\alpha,\ \beta+1;\ z) - (\beta+1)z\,{}_2F_0(\alpha,\ \beta+2;\ z).\qquad(4.3)$$

The ${}_2F_0$ in (4.3) are formal power series. From (4.1) it follows that

$$\psi(z) = \begin{cases} \dfrac{z\,{}_1F_1(b+1;\ c+b+1;\ z)}{{}_1F_1(b;\ c+b;\ z)} & \text{at } 0,\\[2ex] -(c+b)\dfrac{{}_2F_0(-c+1,\ b+1;\ -z^{-1})}{{}_2F_0(-c+1,\ b;\ -z^{-1})} & \text{at } \infty \text{ (formally)}, \end{cases}$$

and

$$\eta(z) = \begin{cases} \dfrac{z\,{}_2F_0(a,\ b+1;\ z)}{{}_2F_0(a,\ b;\ z)} & \text{at } 0 \text{ (formally)},\\[2ex] \dfrac{1}{-a+b+1}\,\dfrac{{}_1F_1(b+1;\ -a+b+2;\ -z^{-1})}{{}_1F_1(b;\ -a+b+1;\ -z^{-1})} & \text{at } \infty. \end{cases}$$

Case (I). Taking $\beta=b+k$ and $\gamma=c+b+k$ in (4.2) we obtain

$$\begin{aligned}{}_1F_1(b+k;\ c+b+k;\ z) &= \left(1-\frac{z}{c+b+k}\right){}_1F_1(b+k+1;\ c+b+k+1;\ z)\\ &\quad+\frac{b+k+1}{(c+b+k)(c+b+k+1)}\\ &\quad\times z\,{}_1F_1(b+k+2;\ c+b+k+2;\ z),\quad k=0,1,\ldots.\end{aligned}$$

This leads to the T-fraction expansion

$$\frac{z\,{}_1F_1(b+1;\ c+b+1;\ z)}{{}_1F_1(b;\ c+b;\ z)} \approx \mathop{\mathrm{K}}_{n=1}^{\infty} \frac{-B(n+b)z}{z-A(n+b)},$$

where now

$$B(k+b) = -(b+k-1), \quad k \geqslant 2, \qquad B(1+b) = c+b,$$
$$A(k+b) = c+b+k-1, \quad k \geqslant 1.$$

It can be shown that also

$$-(c+b)\frac{{}_2F_0(-c+1,\ b+1;\ -z^{-1})}{{}_2F_0(-c+1,\ b;\ -z^{-1})} \approx \mathop{\mathrm{K}}_{n=1}^{\infty} \frac{-B(n+b)z}{z-A(n+b)}.$$

If

$$\mathop{\mathrm{K}}_{k=1}^{n} \frac{-B(k+b)z}{z-A(k+b)} = \frac{P_n(z;\ b)}{Q_n(z;\ b)}, \quad n = 1, 2, \ldots,$$

where P_n and Q_n are polynomials of degree n, Q_n monic, the P_n and Q_n satisfy

$$X_n = (z - A(n+b))X_{n-1} - B(n+b)zX_{n-2}, \quad n = 1, 2, \ldots, \tag{4.4}$$

with

$$P_{-1} = 1, \qquad P_0 = 0, \qquad P_1 = -B(1+b)z, \qquad Q_{-1} = 0, \qquad Q_0 = 1,$$
$$Q_1 = z - A(1+b).$$

Now let

$$q_n(z;\ b) = \lim_{a\to\infty} a^n V_n(a^{-1}z;\ b).$$

Then since

$$\lim_{a\to\infty} {}_4F_3\left(\begin{matrix} b,\ a,\ c+n-k+b,\ -k \\ c+b,\ a-k,\ n-k+b+1 \end{matrix}\middle| 1\right) = {}_3F_2\left(\begin{matrix} b,\ c+n-k+b,\ -k \\ c+b,\ n-k+b+1 \end{matrix}\middle| 1\right),$$

we get

$$q_n(z;\ b) = (-1)^n(c+b)_n \sum_{k=0}^{n} \frac{(-n-b)_k}{k!(-c-n-b+1)_k}$$
$${}_3F_2\left(\begin{matrix} b,\ c+n-k+b,\ -k \\ c+b,\ n-k+b+1 \end{matrix}\middle| 1\right)(-z)^k.$$

From (2.2) we obtain

$$q_n(z;\ b) = (z-(c+b+n-1))q_{n-1}(z;\ b) + (b+n-1)zq_{n-2}(z;\ b), \quad n = 1, 2, \ldots,$$

so $q_n(z;\ b)$ satisfies (4.4). As $q_{-1}(z;\ b) = 0$ and $q_0(z;\ b) = 1$ we have

$$q_n(z;\ b) = Q_n(z;\ b), \quad n = -1, 0, 1, \ldots .$$

Hence, for $n = 0, 1, \ldots$ we have

$$Q_n(z;\ b) = (-1)^n(c+b)_n \sum_{k=0}^{n} \frac{(-n-b)_k}{k!(-c-n-b+1)_k}$$
$${}_3F_2\left(\begin{matrix} b,\ c+n-k+b,\ -k \\ c+b,\ n-k+b+1 \end{matrix}\middle| 1\right)(-z)^k.$$

Since in this case also

$$P_n(z;\ b) = -B(1+b)zQ_{n-1}(z;\ b+1),$$

we have moreover

$$P_n(z;\ b)=(-1)^n(c+b)_n z\sum_{k=0}^{n-1}\frac{(-n-b)_k}{k!(-c-n-b+1)_k}$$
$$\times {}_3F_2\left(\begin{matrix}b+1,\ c+n-k+b,\ -k\\ c+b+1,\ n-k+b+1\end{matrix}\,\middle|\,1\right)(-z)^k.$$

If

$$\psi(z)=\begin{cases}\displaystyle\sum_{n=t}^{\infty}c_n z^n & \text{at } 0,\\ \displaystyle-\sum_{n=0}^{\infty}c_{-n}z^{-n} & \text{at } \infty,\end{cases}$$

then the Laurent polynomials

$$z^{-n}Q_{2n}(z;\ b),\quad z^{-n-1}Q_{2n+1}(z;\ b),\qquad n=0,1,\ldots,$$

form an orthogonal system of Laurent polynomials with respect to the functional Φ given by

$$\Phi(z^n)=c_{-n},\quad n\in\mathbb{Z}.$$

Furthermore P_n/Q_n is the two-point Padé approximant to ψ determined by

$$\psi(z)-\frac{P_n(z;\ b)}{Q_n(z;\ b)}=\begin{cases}\mathrm{O}(z^{n+1}), & z\to 0,\\ \mathrm{O}(z^{-n}), & z\to\infty.\end{cases}$$

Case (II). In this case we take $\alpha=a$ and $\beta=b+k$ in (4.3) to obtain

$${}_2F_0(a,\ b+k;\ z)=(1+(-a+b+k+1)z)\,{}_2F_0(a,\ b+k+1;\ z)$$
$$-(b+k+1)z\,{}_2F_0(a,\ b+k+2;\ z),\quad k=0,1,\ldots,$$

leading to the T-fraction expansion

$$\frac{z\,{}_2F_0(a,\ b+1;\ z)}{{}_2F_0(a,\ b;\ z)}\approx\underset{n=1}{\overset{\infty}{\mathrm{K}}}\frac{-B(n+b)z}{z-A(n+b)},$$

with in this case

$$B(k+b)=\frac{k+b-1}{(-a+k+b-1)(-a+k+b)},\quad k\geqslant 2,\qquad B(1+b)=-\frac{1}{-a+b+1},$$
$$A(k+b)=-\frac{1}{-a+k+b},\quad k\geqslant 1.$$

It can be shown that also

$$\frac{1}{-a+b+1}\,\frac{{}_1F_1(b+1;\ -a+b+2;\ -z^{-1})}{{}_1F_1(b;\ -a+b+1;\ -z^{-1})}\approx\underset{n=1}{\overset{\infty}{\mathrm{K}}}\frac{-B(n+b)z}{z-A(n+b)}.$$

If $R_n(z;\ b)$ and $S_n(z;\ b)$ are polynomials of degree n, $S_n(z;\ b)$ monic, such that

$$\underset{k=1}{\overset{n}{\mathrm{K}}}\frac{-B(k+b)z}{z-A(k+b)}=\frac{R_n(z;\ b)}{S_n(z;\ b)},\quad n=1,2,\ldots,$$

then the R_n and the S_n satisfy

$$X_n = (z - A(n+b))X_{n-1} - B(n+b)zX_{n-2}, \quad n = 1, 2, \ldots,$$

with

$$R_{-1} = 1, \quad R_0 = 0, \quad R_1 = -B(1+b)z, \quad S_{-1} = 0, \quad s_0 = 1,$$
$$S_1 = z - A(1+b).$$

Now from (2.2) we get that the polynomials

$$s_n(z;\ b) = \lim_{c \to \infty} c^{-n} V_n(cz;\ b)$$

satisfy

$$s_n(z;\ b) = \left(z + \frac{1}{-a+n+b}\right) s_{n-1}(z;\ b) - \frac{n+b-1}{(-a+n+b-1)(-a+n+b)} z\, s_{n-2}(z;\ b), \quad n = 1, 2, \ldots,$$

while $s_{-1}(z;\ b) = 0$ and $s_0(z;\ b) = 1$. This means that

$$s_n(z;\ b) = S_n(z;\ b), \quad n = -1, 0, 1, \ldots .$$

Hence calculating $s_n(z; b)$ we get

$$S_n(z;\ b) = \frac{1}{(-a+b+1)_n} \sum_{k=0}^{n} \frac{(-n-b)_k(-a+1)_k}{k!}\ {}_3F_2\left(\begin{matrix} b,\ a,\ -k \\ a-k,\ n-k+b+1 \end{matrix} \middle| 1\right)(-z)^k.$$

From

$$R_n(z;\ b) = -B(1+b)z\, S_{n-1}(z;\ b+1),$$

we finally get

$$R_n(z;\ b) = \frac{1}{(-a+b+1)_n} z \sum_{k=0}^{n-1} \frac{(-n-b)_k(-a+1)_k}{k!}\ {}_3F_2\left(\begin{matrix} b+1,\ a,\ -k \\ a-k,\ n-k+b+1 \end{matrix} \middle| 1\right)(-z)^k.$$

The Laurent polynomials

$$z^{-n} S_{2n}(z;\ b), \quad z^{-n-1} S_{2n+1}(z;\ b), \quad n = 0, 1, \ldots,$$

are orthogonal with respect to the moments generated by η, and R_n/S_n is the two-point Padé approximant to η with

$$\eta(z) - \frac{R_n(z;\ b)}{S_n(z;\ b)} = \begin{cases} O(z^{n+1}), & z \to 0, \\ O(z^{-n}), & z \to \infty. \end{cases}$$

Remark 4.1. Writing $\psi(b, c, z)$, $\eta(a, b; z)$, $P_n(z; b, c)$, $Q_n(z; b, c)$, $R_n(z; a, b)$, $S_n(z; a, b)$ for ψ, η, P_n, Q_n, R_n, S_n, respectively, we have

$$(-a+b+1)z\ \eta(a,\ b;\ -z^{-1}) = \psi(b,\ -a+1;\ z).$$

and from

$$\eta(a, b; z) - \frac{R_n(z; a, b)}{S_n(z; a, b)} = \begin{cases} O(z^{n+1}), & z \to 0 \\ O(z^{-n}), & z \to \infty, \end{cases}$$

we thus obtain

$$\psi(b, -a+1; z) - (-a+b+1)\frac{z^{n+1}R_n(-z^{-1}; a, b)}{z^n S_n(-z^{-1}; a, b)} = \begin{cases} O(z^{n+1}), & z \to 0, \\ O(z^{-n}), & z \to \infty. \end{cases}$$

Since

$$(-a+b+1)_n z^n S_n(-z^{-1}; a, b)$$

is monic, we get

$$Q_n(z; b, -a+1) = (-a+b+1)_n z^n S_n(-z^{-1}; a, b) \tag{4.5}$$

and a similar result for the P_n and the R_n. Comparing coefficients of z^k in both sides of (4.5) gives

$$\frac{(-n-b)_{n-k}(-a+1)_{n-k}}{(n-k)!}\, {}_3F_2\left(\begin{matrix} b,\ a,\ -n+k \\ a-n+k,\ k+b+1 \end{matrix}\middle| 1\right)$$
$$= \frac{(-1)^{n-k}(-a+b+1)_n(-n-b)_k}{k!(a-n-b)_k}$$
$$\times {}_3F_2\left(\begin{matrix} b,\ -a+n-k+b+1,\ -k \\ -a+b+1,\ n-k+b+1 \end{matrix}\middle| 1\right), \quad k=0, 1,\ldots, n, \quad n=0, 1, 2,\ldots .$$

Is this a well-known relation between terminating ${}_3F_2$ with unit argument?

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